ROBUST STABILIZATION OF UNCERTAIN, UNSTABLE PERIODIC ORBITS USING GENERALIZED SAMPLED-DATA HOLD FUNCTION CONTROL

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The problem of robust stabilization of unstable periodic solutions of nonlinear dynamic systems subject to parametric uncertainties is treated. The challenge is to achieve robust stability of orbital motion when both the period and the location of the periodic orbit are uncertain. The technical approach used here is to extend the method of Generalized Sampled-data Hold Function control of linear periodic systems in two ways. First, a change of independent variable is introduced, from time to a new angular variable, making the control logic event-driven rather than time-driven. Second, an adaptation strategy is introduced, whereby the uncertain parameters are estimated online, the reference periodic orbit is computed based on the parameter estimates, and event-driven control is implemented to stabilize the motion about the parameter-dependent reference periodic orbit. The robustness of this adaptive event-driven strategy is illustrated by the robust stabilization of an unstable Halo orbit in a general planet-satellite system under uncertainty of the mass parameter.

INTRODUCTION

This paper is devoted to the problem of robust stabilization of unstable periodic solutions in nonlinear dynamic systems possessing parametric uncertainties. Local stabilization of a nonlinear system about a given unstable periodic orbit can be in general achieved by stabilization of a linear periodic system obtained by linearization about a periodic solution. Stabilization of linear periodic systems can be achieved by monodromy matrix assignment using Generalized Sampled-data Hold Function (GSHF) control techniques (See Ref. 1). In this paper we consider uncertain parameters in the nonlinear system and present extensions of the GSHF control method to address robust stabilization of systems about unstable periodic orbits subject to parametric uncertainties. Possible applications of the theory include stabilization of spacecraft flight about unstable Halo orbits and active stabilization of gaits for walking robots.

When a nonlinear dynamic system has a periodic orbit, a parametric perturbation of the system may have two consequences. First, the period of the orbit may change, making periodic control strategies with fixed period ineffective. Second, the periodic orbit itself may move, defeating control strategies based on linearization about a fixed reference orbit. Therefore, robust stabilization of periodic orbits in nonlinear systems requires addressing these two issues, and this is the technical contribution of this paper.

Since in a nonlinear autonomous system an equilibrium point can be viewed as a trivial periodic solution, the problem of stabilizing uncertain periodic orbits is related to that of stabilizing uncertain equilibrium points. The latter problem has been considered in Refs. 3, 4. Specifically, Ref. 3 uses an equilibrium estimator and gain scheduling techniques for tracking the uncertain equilibrium, while

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Ref. 4 uses the concept of parametric stability and combines optimization techniques to produce a robust control law that accounts for uncertainty of the equilibrium. In spite of their usefulness for uncertain equilibria, the results in Refs. 3, 4 are not directly applicable to nontrivial periodic solutions.

References 5, 6 present robust controllers for uncertain linear periodic systems. As it was noted above, robust stabilization of periodic orbits in nonlinear systems requires accounting for two phenomena: changes in orbital period and changes in orbit location. Because of these two issues, robustness of periodic solutions in nonlinear systems is different from robustness of linear periodic systems. The topic considered in this paper is the former, while the topic considered in Refs. 5, 6 is the latter. To our knowledge there is no work in the literature specifically addressing robust stabilization of periodic orbits in nonlinear systems.

The idea behind GSHF control is very simple: sample the output of the system at the beginning of each period and, during the next period, generate the control input by modulating the sampled output using a hold function. In practice, GSHF control is implemented as continuous-time linear periodic output feedback. In this paper, the method of GSHF control is extended in two ways, to make it applicable to the robust stabilization of uncertain periodic orbits by addressing the two issues of period change and orbit change. First, a change of independent variable is introduced, from time $t$ to a new angular variable $\theta$, turning the control strategy into “Event-Driven” GSHF control, rather than the original “Time-Driven” GSHF control. With the new independent variable, the period of the orbit becomes independent of any uncertain parameters and is always $2\pi$. The second extension of GSHF control introduces an adaptation strategy, whereby the uncertain parameters are estimated online, the reference periodic orbit is computed based on the parameter estimates, and ED-GSHF control is implemented to stabilize the motion about the parameter-dependent reference periodic orbit. Hence, from the standpoint of theory, the original contributions of this paper consist of these two extensions of the method of GSHF control; and, from the standpoint of application, the original contribution is in the robust stabilization of uncertain, unstable periodic orbits.

In order to illustrate the control method we apply it to stabilization of spacecraft flight about an unstable Halo orbit. Halo orbits are potentially unstable periodic orbits that usually exist in the vicinity of the libration points of the restricted circular three-body problem (See Refs. 7–9). In spite of their potential instability, they are often appealing for space flight: for example, those near the Sun-Earth Lagrangian points are, because of their uniquely opportune locations, ideal for long-term Sun, stellar, or Earth observations (See Ref. 10). Stabilization of motion about Halo orbits has been studied in Refs. 11–13. Reference 11 develops a control law based on modal transformation and Floquet theory, and Ref. 12 presents a control method that achieves neutral stability by eliminating hyperbolic manifolds at each instant while preserving a Hamiltonian structure. Reference 13 applies $\mathcal{H}_\infty$ techniques for linear time-varying systems to achieve asymptotic stabilization. References 14–17, on the other hand, investigate station keeping algorithms near unstable Halo orbits. None of the above references, however, consider parametric uncertainty in dynamics and robustness of stabilizing or station keeping controls. In this paper, we use Hill’s equations of motion as the model of spacecraft flight and assume uncertainty in primary mass parameter. Our goal is then to achieve robust stabilization about an unstable Halo orbit with respect to the uncertain mass parameter.

The remainder of the paper is as follows. In the next section we provide some mathematical preliminaries and state formally the problems considered. First, we present our results on Time-Driven GSHF control of nonlinear systems and illustrate its lack of robustness on a simple example. Next, we present Event-Driven GSHF control to address the period uncertainty and show that the robustness is improved compared to Time-Driven GSHF control. Next, we present Adaptive Event-Driven GSHF control, which addresses parameter-dependent orbit changes through estimation. Two different adaptive strategies are presented. We also derive the first-order approximation of perturbed periodic orbits that may be used together with the adaptive strategies. The reversed-time Van der Pol system, which has an unstable periodic orbit, is used throughout to illustrate the technical points.
Finally, we discuss application of Adaptive Event-Driven GSHF control to robust stabilization of a Halo orbit. We simulate and compare controlled trajectories obtained by Time-Driven GSHF control and Adaptive Event-Driven GSHF control. In the tutorial Appendix, we summarize basic results on GSHF control of linear periodic systems.

**MATHEMATICAL PRELIMINARIES AND PROBLEM STATEMENTS**

**Systems Without Uncertainty**

Consider the nonlinear system

\[ \dot{x} = f(x, u), \]  

where \( x(t) \in D \subset \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^r \) is the control input, and the function \( f \) is analytic with respect to its arguments. We call a solution of System (1) with no control, i.e., with \( u(t) \equiv 0 \), a *natural solution* and its path, i.e., the locus in the state space of the solution \( x(t) \) for all \( t \geq 0 \), a *natural orbit*. Let \( x(t) = \phi(t, x_0) \) denote the natural solution of System (1) with initial state \( x(0) = x_0 \). Since \( f \) is analytic, \( \phi(t, x_0) \) exists and is unique locally. A nonconstant natural solution \( \phi(t, x_0^*) \) satisfying

\[ \phi(t + T, x_0^*) = \phi(t, x_0^*), \]  

for some \( T > 0 \) and for all \( t \geq 0 \) is called a *natural periodic solution* and its path, which is a simple closed curve in the state space, is called a *natural periodic orbit*, while \( T \) is the period of the periodic orbit. Note that if \( \mathcal{C} \) is a natural periodic orbit, then for any \( y \in \mathcal{C} \), \( \phi(t, y) \) is a natural periodic solution.

For later use, we review two different notions of stability for solutions of nonlinear autonomous systems:

**Definition 1** (Orbital Stability\(^{18}\)). Let \( \mathcal{H}^* \) be the path of the solution \( x^*(t) \) of the autonomous system \( \dot{x} = f(x) \), which starts at \( a^* \) at \( t = t_0 \). Suppose that for every \( \varepsilon > 0 \) there exist \( \delta(\varepsilon) > 0 \) such that if \( \mathcal{H} \) is the path starting at \( a \),

\[ |a - a^*| < \delta \Rightarrow \max_{x \in \mathcal{H}} \text{dist}(x, \mathcal{H}^*) < \varepsilon, \]

where the distance from a point \( x \) to a curve \( \mathcal{H} \) is defined by

\[ \text{dist}(x, \mathcal{H}) = \min_{y \in \mathcal{H}} |x - y|. \]

Then \( \mathcal{H}^* \) is said to be orbitally stable, or Poincaré stable. If \( \mathcal{H}^* \) is orbitally stable and

\[ \lim_{t \to \infty} \text{dist}(x(t), \mathcal{H}^*) = 0 \]

whenever \( x(t_0) \) belongs to a neighborhood of \( a^* \), then \( \mathcal{H}^* \) is said to be orbitally asymptotically stable.

**Definition 2** (Lyapunov Stability\(^{18}\)). Let \( x^*(t) \) be a given solution of the autonomous system \( \dot{x} = f(x) \). Then \( x^*(t) \) is said to be Lyapunov stable if and only if for every \( \varepsilon > 0 \), there exist \( \delta(\varepsilon) > 0 \) such that

\[ |x(t_0) - x^*(t_0)| < \delta \Rightarrow |x(t) - x^*(t)| < \varepsilon \]

for all \( t > t_0 \). If \( x^*(t) \) is Lyapunov stable and \( \lim_{t \to \infty} |x(t) - x^*(t)| = 0 \) whenever \( x(t_0) \) belongs to a neighborhood of \( x^*(t_0) \), then \( x^*(t) \) is said to be Lyapunov asymptotically stable.

Orbital stability is a relatively mild stability criterion compared to Lyapunov stability. Specifically, Lyapunov stability of a solution implies orbital stability of the path of the solution for autonomous systems.\(^{18}\) Although the notions of stability are defined for orbits or solutions and not
for systems, we use in places the words informally, such as “the system \( \dot{x} = f(x) \) is orbitally stable about \( \mathcal{X}^* \),” the meaning of which is quite clear.

The above stability definitions for general orbits and solutions of autonomous systems can be directly applied to natural periodic orbits and solutions of System (1). The interpretation of stability for natural periodic orbits is obvious: If \( \mathcal{C} \) is an orbitally stable (resp., orbitally asymptotically stable) natural periodic orbit, then all solutions starting in the vicinity of \( \mathcal{C} \) stay close to \( \mathcal{C} \) (resp., stay close to and converge to \( \mathcal{C} \)) without control. But if \( \mathcal{C} \) is an orbitally unstable periodic orbit, then a solution starting arbitrarily close to \( \mathcal{C} \) escapes the vicinity of \( \mathcal{C} \) when no control is applied. We can now state the first problem treated in this paper as follows.

**Problem 1 (Nominal Stabilization).** Suppose that System (1) has an orbitally unstable periodic orbit \( \mathcal{C} \) with period \( T \). Design a control \( u(t, x) \) that locally achieves orbital asymptotic stability of System (1) about \( \mathcal{C} \).

**Systems With Uncertainty**

Consider now the nonlinear system containing uncertain parameters

\[
\dot{x} = f(x, u, p),
\]

where \( x(t) \) and \( u(t) \) are as in (1), \( p \in P \subset \mathbb{R}^n \) is a vector of constant parameters, and the function \( f \) is analytic with respect to its arguments. We assume that the value of \( p \) is uncertain but close to a nominal value \( \bar{p} \in P \). Let \( x(t) = \phi(t, x_0, p) \) denote the natural solution of System (3) with initial state \( x(0) = x_0 \). A natural periodic solution \( \phi(t, x_0^*, p) \) has to satisfy, for some \( T > 0 \),

\[
\phi(t + T, x_0^*, p) = \phi(t, x_0^*, p), \quad \text{for all } t \geq 0,
\]

or

\[
\phi(T, x_0^*, p) - x_0^* = 0.
\]

Suppose that System (3) with the nominal parameter value, i.e., with \( p = \bar{p} \), has a natural periodic orbit \( \mathcal{C} \) with period \( \bar{T} \). In general, if \( p \neq \bar{p} \), then \( \mathcal{C} \) is no longer a natural periodic orbit of System (3). However, if \( p \) is sufficiently close to \( \bar{p} \), then a natural periodic orbit of System (3) may exist in the vicinity of \( \mathcal{C} \). Indeed, for \( x_0^* \in \mathcal{C} \), consider a hyperplane in \( \mathbb{R}^n \) passing through \( x_0^* \) without being tangent to \( \mathcal{C} \) at \( x_0^* \) and let \( x_0^* \) be constrained to belong to this hyperplane\(^a\), then

\[
c^T(x_0^* - x_0^*) = 0,
\]

for a nonzero constant vector \( c \in \mathbb{R}^n \). Now, (5)-(6) are a system of \( n + 1 \) equations for the \( n + 1 \) unknowns \((x_0^*, T)\). The Implicit Function Theorem\(^b\) guarantees that if the Jacobian determinant of the left-hand sides of (5)-(6) with respect to \((x_0^*, T)\) is nonzero at \((x_0^*, \bar{T}, \bar{p})\) then there is a function \((x_0^*(p), T(p))\) of \( p \), defined in a neighborhood of \( \bar{p} \), such that \((x_0^*(p), T(p))\) satisfies (5)-(6) identically, or in other words, such that \( \phi(t, x_0^*(p), p) \) is a natural periodic solution of (3) with period \( T(p) \).

Let us assume the existence of this function \((x_0^*(p), T(p))\) and let \( \mathcal{C}(p) \) denote the periodic orbit associated with the natural periodic solution \( \phi(t, x_0^*(p), p) \). We refer to \( \bar{\mathcal{C}} \) and \( \phi(t, x_0^*(\bar{p}), \bar{p}) \) as the nominal natural periodic orbit and solution, respectively, and \( \mathcal{C}(p) \) and \( \phi(t, x_0^*(p), p) \) as the true natural periodic orbit and solution, respectively. Figure 1 illustrates schematically the hyperplane constraint on the initial state and the shift of periodic orbit due to perturbation of \( p \) from \( \bar{p} \).

**Remark 1.** Note that even if the periodic orbits \( \bar{\mathcal{C}} \) and \( \mathcal{C}(p) \) are relatively close to each other when \( p \) is close to \( \bar{p} \), the values of \( \phi(t, x_0^*(\bar{p}), \bar{p}) \) and \( \phi(t, x_0^*(p), p) \) can be significantly different at some \( t \) because the periods of the two orbits, \( \bar{T} \) and \( T(p) \), are generally different.

\(^a\)More generally, a hypersurface may be used instead of a hyperplane.
We can now state the second problem treated in this paper as follows.

**Problem 2 (Robust Stabilization).** Suppose that for \( p = \bar{p} \), System (3) has an orbitally unstable natural periodic orbit \( \bar{C} \) with period \( \bar{T} \). Design a control \( u(t, x) \) that locally achieves robust orbital asymptotic stability of System (3) about its natural periodic orbit \( \bar{C}(p) \), i.e., that achieves orbital asymptotic stability whenever \( p \) is sufficiently close to \( \bar{p} \).

**TIME-DRIVEN GSHF CONTROL**

If a nonlinear system is linearized about a periodic solution, we obtain a linear periodic system. The periodic solution can be locally stabilized by stabilizing the linear periodic system. It is well known that Generalized Sampled-data Hold Function (GSHF) control can be used to stabilize linear periodic systems (See Ref. 1). The techniques of GSHF control on linear periodic systems are reviewed in Appendix. The first approach to Problem 1 is a direct application of the GSHF control techniques.

Assume that the initial state \( x(0) \) is sufficiently close to the natural periodic orbit \( \bar{C} \) of System (1). First, choose a point \( x_0^* \in \bar{C} \) which is sufficiently close to \( x(0) \). Next, take the natural periodic solution \( \phi(t, x_0^*) \) as the reference solution and linearize System (1) about it. We then obtain the following linear system:

\[
\delta x = A(t)\delta x + B(t)u,
\]

where \( \delta x(t) = x(t) - \phi(t, x_0^*) \) and

\[
A(t) = \frac{\partial f}{\partial x}(\phi(t, x_0^*), 0), \quad B(t) = \frac{\partial f}{\partial u}(\phi(t, x_0^*), 0).
\]

Since \( \phi(t, x_0^*) \) is periodic with period \( T \), the matrices \( A \) and \( B \) are periodic with period \( T \), i.e., \( A(t + T) = A(t), B(t + T) = B(t) \) for all \( t \geq 0 \), making System (7) a linear periodic system. We can design a GSHF control \( u \) that stabilizes System (7) using the technique in Appendix. Let \( m(M) \) be some arbitrary measure of singularity for a square matrix \( M \), such as the condition number of \( M \), and let \( m_0 \) be a threshold such that for \( M \) with \( m(M) \leq m_0 \) the inverse \( M^{-1} \) is well numerically computed. The proposed control law \( u(t, x) \) is given as a function of \( t \) and \( x \) as follows:

\[
u(kT + \sigma) = \begin{cases} H(\sigma)\Psi(\sigma)^{-1}\{x(kT + \sigma) - \phi(\sigma, x_0^*)\}, & \text{if } m(\Psi(\sigma)) \leq m_0; \\ H(\sigma)\Psi(\sigma_M)^{-1}\{x(kT + \sigma_M) - \phi(\sigma_M, x_0^*)\}, & \text{if } m(\Psi(\sigma)) > m_0, \end{cases}
\]

\[
\sigma_M = \max\{t \in [0, \sigma) : m(\Psi(t)) \leq m_0\},
\]

where \( \Psi(\sigma) \) is some function of \( \sigma \) and \( \Psi(0) = I \) (the identity matrix).

**Figure 1:** Shift of periodic orbit due to parameter perturbation; the period also changes in general.
for $\sigma \in [0, T)$ and $k = 0, 1, 2, \ldots$, where

$$H(t) = B(t)^T \Phi(t, t)^T K,$$

$$\Psi(t) = \Phi(t, 0) + W(t)K,$$

where $\Phi(t, t_0)$ and $W(t)$ are obtained by

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I,$$

$$W(t) = \int_0^t \Phi(t, \tau)B(\tau)B(\tau)^T \Phi(t, \tau)^T d\tau, \quad t \in [0, T],$$

and

$$K = -\left((W(T)^T SW(T) + R')^{-1} (W(T)^T S\Phi(T, 0) + V')\right),$$

where $S$ is the positive semi-definite solution of the Algebraic Riccati Equation

$$\Phi(T, 0)^T S\Phi(T, 0) - S + Q' - \left((\Phi(T, 0)^T SW(T) + V') (W(T)^T SW(T) + R')^{-1} (W(T)^T S\Phi(T, 0) + V')\right) = 0,$$

for user specified weighting matrices $Q'$, $R'$, and $V'$. It is remarked that GSHF control is implemented as continuous feedback control whenever $\Psi(\sigma)^{-1}$ is available (the first line of (9)) to cope with nonlinear deviations. When $\Psi(\sigma)^{-1}$ is unavailable the state $x(kT + \sigma_M)$ is held until $\Psi(\sigma)^{-1}$ becomes available next time (the second line of (9)). It is sometimes necessary to scale the equation (7) so that $\Psi(\sigma)^{-1}$ is available most of the time in $[0, T)$.

We refer to the matrices $\Phi(T, 0)$ and $\Psi(T)$ in (12) as the open-loop and closed-loop monodromy matrices, respectively. These matrices indicate how the natural and controlled dynamics evolve at the sampling instants $t = kT$, $k = 0, 1, 2, \ldots$, respectively. The Lyapunov asymptotic stability of the natural and controlled systems are equivalent to the requirements that all the eigenvalues of $\Phi(T, 0)$ and $\Psi(T)$ have moduli strictly less than 1, respectively. If the pair $(\Phi(T, 0), W(T))$ is stabilizable and the pair

$$(Q' - V'R'^{-1}V'^T, \Phi(T, 0) - W(T)R'^{-1}V'^T)$$

has no unobservable mode on the unit circle, then the control (9) makes the linear system (7) asymptotically stable and in turn locally achieves Lyapunov asymptotic stability of System (1) about $\phi(t, x_n^0)$, which implies orbital asymptotic stability of System (1) about $\mathcal{C}$ as desired. We call the control (9) Time-Driven GSHF control since its logic is time-driven, or specifically it uses the idea of sampling the state according to the time-period of the reference periodic orbit.

**Example 1** (Reversed-Time Van der Pol System). The standard Van der Pol nonlinear system is known to have a stable limit cycle.\(^{20}\) In this example, we consider the system obtained by reversing time in the standard Van der Pol system and also adding control inputs:

$$\begin{align*}
\dot{x} &= -y + u_x, \\
\dot{y} &= x - p(1 - x^2)y + u_y,
\end{align*}$$

where $[x, y]^T$ is the state, $[u_x, u_y]^T$ is the control input, and $p$ is a constant parameter. Let the nominal parameter value be $\bar{p} = 0.6$. System (17) has an unstable natural periodic orbit shown in Figure 2 for $p = \bar{p}$.

We shall nominally stabilize System (17) for $p = \bar{p}$ about the natural periodic orbit in Figure 2 using Time-Driven GSHF control. The control law is synthesized according to (9)-(16), where the weighting matrices are chosen as $Q' = I$, $R' = I$, and $V' = 0$. The eigenvalues of the open-loop and closed-loop monodromy matrices are obtained as $\{51.35, 1.000\}$ and $\{0.0659, 7.25 \times 10^{-6}\}$,
respectively. A controlled trajectory of the nominal system with Time-Driven GSHF control was simulated and is shown in Figure 3. The initial state for the simulation was set off the periodic orbit by a distance of 0.3 in the radial direction. It is seen that the controlled trajectory stays close to and converges to the natural periodic orbit and therefore nominal asymptotic stability about the orbit is achieved by Time-Driven GSHF control.

Next, in order to assess the robustness of the controller with respect to parameter perturbation, we simulated the closed-loop system with $p = \bar{p} = 0.8$, i.e., where the parameter value is perturbed by $+0.2$, using the same controller as in the previous simulation. The simulated controlled trajectory is shown in Figure 4. It is seen that, as a result of the parameter perturbation, the controller fails to keep the controlled trajectory in the vicinity of the periodic orbit. Figure 5 compares the natural periodic orbits of the Reversed-Time Van der Pol system for the two different parameter values, $p = 0.6$ and $p = 0.8$. Note that the periods of the two periodic orbits are also different. This makes the formal stability analysis of the perturbed system difficult as we explain below: the dynamics are periodic with period 6.53, whereas the controller is periodic with period 6.42.

Motivated by the example above, let us next investigate the robustness of Time-Driven GSHF controllers with respect to parameter perturbations. Consider System (3). Suppose that we have
designed a Time-Driven GSHF controller

\[ u(t) = G(t) \{x(t) - \phi(t, x_0^*(p), \bar{p})\}, \quad G(t + \bar{T}) = G(t), \]  

(18)

which nominally stabilizes System (3) for \( p = \bar{p} \) about the nominal periodic orbit \( \bar{C} \). Apply this control to System (3) and linearize the system about the true periodic solution \( \phi(t, x_0^*(p), p) \), then we obtain

\[
\delta x = \left[ A(t, p) + B(t, p)G(t) \right] \delta x + G(t) \left\{ \phi(t, x_0^*(p), p) - \phi(t, x_0^*(\bar{p}), \bar{p}) \right\},
\]

where \( \delta x(t) = x(t) - \phi(t, x_0^*(p), p) \) and

\[
A(t, p) = \frac{\partial f}{\partial x}(\phi(t, x_0^*(p), p), 0, p), \quad B(t, p) = \frac{\partial f}{\partial u}(\phi(t, x_0^*(p), p), 0, p). \]  

(20)

A formal stability analysis of System (19) is difficult because the matrices \( A(t, p) \) and \( B(t, p) \) are periodic with period \( T(p) \), whereas the matrix \( G(t) \) is periodic with period \( T, T(p) \neq T \) and generally \( T(p) \) and \( T \) are not commensurate. Nonetheless, we see that the parameter error, i.e., \( p \neq \bar{p} \), affects the closed-loop system in two ways: error in the dynamics and error in the reference, the latter of which enters the linearized system as a disturbance term. In order for the controlled orbit to stay close to \( \mathcal{C}(p) \), the linear system (19) has to be stable (it is assumed to be stable with \( \bar{x}, \bar{T}, \bar{p} \)) and generally \( \bar{x} \) and \( \bar{T} \) are not commensurate. Nonetheless, we see that the parameter error, i.e., \( p \neq \bar{p} \), affects the closed-loop system in two ways: error in the dynamics and error in the reference, the latter of which enters the linearized system as a disturbance term. In order for the controlled orbit to stay close to \( \mathcal{C}(p) \), the linear system (19) has to be stable (it is assumed to be stable with \( \bar{p} \)) and the disturbance term in (19) has to be sufficiently small. But since values of \( \phi(t, x_0^*(\bar{p}), \bar{p}) \) and \( \phi(t, x_0^*(p), p) \) can be significantly different due to period difference as noted in Remark 1, \( A(t, p) \) and \( B(t, p) \) can be significantly different from their nominal values \( A(t, \bar{p}) \) and \( B(t, \bar{p}) \), and the disturbance term may be unacceptably large even if \( p \) is relatively close to \( \bar{p} \). Therefore, the above analysis suggests that Time-Driven GSHF controllers may generally be very sensitive to parameter perturbations.

**EVENT-DRIVEN GSHF CONTROL**

The second approach to Problem 1 is a more sophisticated application of GSHF control and aims at overcoming the robustness deficiency of Time-Driven GSHF control due to uncertainty of the period. The key idea is to change the independent variable from time \( t \) to a new angular variable \( \theta \). This way, we can make the period of a periodic orbit constant with respect to the uncertain parameter vector \( p \).

Consider System (1). For \( y \in \mathcal{C} \), define a smooth map \( \alpha : D \to \mathbb{R}^2 \) such that \( h(t) \triangleq \alpha(\phi(t, y)) \) is a regular closed (parameterized) curve of period \( T \) on \( \mathbb{R}^2 \), whose interior includes the origin \( O \), and its direction is strictly counter-clockwise with respect to \( O \). For a solution \( x(t) \) of System (1), let \( X \) represent the point at \( \alpha(x(0)) \) and \( P \) represent the point at \( \alpha(x(t)) \). Then, let \( \theta(t) \) be the rotational angle of radius \( OP \) starting from radius \( OX \) (see Figure 6). We assume \( d\theta/dt > 0 \) for all \( t \), which is a reasonable assumption because \( \alpha(x(t)) \) is expected to be close to \( \alpha(\mathcal{C}) \) due to the stabilizing control. Note that \( \theta(t = 0) = 0 \).

We change the independent variable of the system from \( t \) to \( \theta \). Transform (1) as

\[ \frac{dx}{d\theta} = \left( \frac{d\theta}{dt} \right)^{-1} f(x, u). \]  

(21)

If we write \( v = [v_1, v_2]^T = \alpha(x(t)) \), then

\[ \frac{d\theta}{dt} = \frac{\partial \theta}{\partial v} \frac{\partial v}{\partial x} \frac{dx}{dt} = \frac{\partial \theta}{\partial v} \frac{\partial v}{\partial x} \underbrace{f(x, u)}_{1 \times 2 \ 2 \times n \ n \times 1}. \]

\(^b\)Actually, the following analysis is also valid for any time-periodic gain feedback controller as the form of (18) suggests.
where $\partial \theta / \partial v$ and $\partial \alpha / \partial x$ are Jacobian matrices. Therefore, we obtain

$$\frac{dx}{d\theta} = \left( \frac{\partial \theta}{\partial v} \frac{\partial \alpha}{\partial x} f(x, u) \right)^{-1} f(x, u) \triangleq \tilde{f}(x, u).$$

(22)

Note that since $\tan(\theta + \theta_0) = v_2 / v_1$, where $\theta_0$ is a constant, we have

$$\frac{\partial \theta}{\partial v_1} = -\frac{v_2}{v_1^2 + v_2^2}, \quad \frac{\partial \theta}{\partial v_2} = \frac{v_1}{v_1^2 + v_2^2}.$$  

(23)

Equation (22) is equivalent to (1) as long as $d\theta/dt > 0$, for all $t \geq 0$, which is assumed to be true at least in a neighborhood of $C$. Thus, in a neighborhood of $C$ we have the system

$$x' = \tilde{f}(x, u),$$  

(24)

where a prime denotes $d/d\theta$. A solution $x(\theta)$ of System (24) is given as a function of $\theta$. A natural solution of System (24), i.e., with $u \equiv 0$, and the initial state $x(0) = x_0$ is denoted by $\tilde{\phi}(\theta, x_0)$. Note that $\tilde{\phi}(\theta, x_0)$ is simply a reparameterization of $\phi(t, x_0)$ and the two have the same path. Consequently, if $y \in \mathcal{C}$, then $\tilde{\phi}(\theta, y)$ is periodic and its orbit is $\mathcal{C}$. Also note that the period of the periodic solution $\tilde{\phi}(\theta, y)$ is always $2\pi$ regardless of the original time period $T$ of $\phi(t, y)$.

Now, let $z \in \mathbb{R}^2$ be the point where the radius $OX$ and the closed curve $\alpha(\mathcal{C})$ intersect. Then, let $x_0^* = \{ y \in \mathcal{C} : \alpha(y) = z \}$. This way, we obtain the reference natural periodic solution $\tilde{\phi}(\theta, x_0^*)$ such that $\alpha(x(\theta))$ and $\alpha(\tilde{\phi}(\theta, x_0^*))$ are on the same radius for all $\theta \geq 0$. Note that since the map $\alpha$ is smooth, relative proximity of two points is preserved through this mapping. Linearize System (24) about $\tilde{\phi}(\theta, x_0^*)$ to obtain the following linear system:

$$\delta x' = A(\theta) \delta x + B(\theta) u,$$  

(25)

where $\delta x(\theta) = x(\theta) - \tilde{\phi}(\theta, x_0^*)$ and

$$A(\theta) = \frac{\partial \tilde{f}}{\partial x}(\tilde{\phi}(\theta, x_0^*), 0), \quad B(\theta) = \frac{\partial \tilde{f}}{\partial u}(\tilde{\phi}(\theta, x_0^*), 0).$$  

(26)

Since $\tilde{\phi}(\theta, x_0^*)$ is periodic with period $2\pi$, so are the matrices $A$ and $B$, i.e., $A(\theta + 2\pi) = A(\theta)$, $B(\theta + 2\pi) = B(\theta)$ for all $\theta \geq 0$, making System (25) a linear periodic system. We can design a GSHF control that stabilizes System (25) using the technique in Appendix. The symbols $t$ and $T$
in Appendix are replaced by \( \theta \) and \( 2\pi \), respectively. The proposed control law \( u(\theta, x) \) is given as a function of \( \theta \) and \( x \) as follows:

\[
u(\theta = 2\pi k + \sigma) = \begin{cases} H(\sigma)\Psi(\sigma)^{-1}\{x(2\pi k + \sigma) - \tilde{\phi}(\sigma, x_0^*)\}, & \text{if } m(\Psi(\sigma)) \leq m_0; \\ H(\sigma)\Psi(\sigma_M)^{-1}\{x(2\pi k + \sigma_M) - \tilde{\phi}(\sigma_M, x_0^*)\}, & \text{if } m(\Psi(\sigma)) > m_0, \end{cases}
\]

(27)

\( \sigma_M = \max\{\theta \in [0, \sigma) : m(\Psi(\theta)) \leq m_0\} \),

(28)

for \( \sigma \in [0, 2\pi) \) and \( k = 0, 1, 2, \ldots \), where

\( H(\theta) = B(\theta)^T\Phi(2\pi, \theta)^TK, \)

(29)

\( \Psi(\theta) = \Phi(\theta, 0) + W(\theta)K, \)

(30)

where \( \Phi(\theta, \theta_0) \) and \( W(\theta) \) are obtained by

\[
\Phi'(\theta, \theta_0) = A(\theta)\Phi(\theta, \theta_0), \quad \Phi(\theta, \theta_0) = I,
\]

(31)

\[
W(\theta) = \int_0^\theta \Phi(\theta, \tau)B(\tau)^T\Phi(2\pi, \tau)^T\,d\tau, \quad \theta \in [0, 2\pi],
\]

(32)

and

\[
K = -\left(W(2\pi)^T S W(2\pi) + R^\prime\right)^{-1}\left(W(2\pi)^T S \Phi(2\pi, 0) + V^\prime\right),
\]

(33)

where \( S \) is the positive semi-definite solution of the Algebraic Riccati Equation

\[
\Phi(2\pi, 0)^T S \Phi(2\pi, 0) - S + Q^\prime - \left(W(2\pi)^T S \Phi(2\pi, 0) + V^\prime\right)^{-1}\left(W(2\pi)^T S \Phi(2\pi, 0) + V^\prime\right) = 0,
\]

(34)

for user specified weighting matrices \( Q^\prime, R^\prime, \) and \( V^\prime \), such that \( R^\prime > 0 \) and \( Q^\prime - V^\prime R^\prime^{-1}V^\prime \geq 0 \).

Similar to the case of Time-Driven GSHF control, we refer to the matrices \( \Phi(2\pi, 0) \) and \( \Psi(2\pi) \) as the open-loop and closed-loop monodromy matrices, respectively. Here also, the locations of the eigenvalues of these matrices with respect to the open unit disk determine the Lyapunov asymptotic stability of the natural and controlled system, respectively. If the pair \( (\Phi(2\pi, 0), W(2\pi)) \) is stabilizable and the pair

\[
(Q^\prime - V^\prime R^\prime^{-1}V^\prime, \Phi(2\pi, 0) - W(2\pi)R^\prime^{-1}V^\prime)
\]

has no unobservable mode on the unit circle, then the control (27) makes the linear system (25) asymptotically stable and in turn locally achieves Lyapunov asymptotic stability of System (25) about \( \tilde{\phi}(\theta, x_0^*) \), which implies orbital asymptotic stability of System (1) about \( \mathcal{C} \) as desired. However, since \( \tilde{\phi}(\tilde{\theta}(t), x_0^*) \neq \phi(t, x_0^*) \) in general, Lyapunov stability with respect to \( \phi(t, x_0^*) \) is not guaranteed. The control \( u(\theta) \) depends explicitly on \( \theta \), but \( \theta \) (more precisely \( \sigma = \theta - 2\pi k \) in (27)) is determined by the state \( x(t) \) (by measuring \( \angle XOP \)). Therefore, this control is given as a function of only the state. We call the control in (27) Event-Driven GSHF control since its logic is event-driven, or specifically, it uses the idea of sampling the state according to the “event” that the state passes the Poincaré surface \( \theta = 2\pi k \).

Let us investigate the robustness of Event-Driven GSHF controllers with respect to parameter perturbations. Consider System (3). The mapping \( \alpha \) is defined based on the nominal periodic orbit \( \mathcal{C} \) in this case. By changing the independent variable from \( t \) to \( \theta \), System (3) is transformed into

\[
\dot{x} = f(x, u, p).
\]

(35)
Let $\tilde{\phi}(\theta, x_0, p)$ denote a natural solution of System (35) starting at $x_0$. Suppose that we have designed the Event-Driven GSHF controller

$$u(\theta) = G(\theta) \left\{ x(\theta) - \tilde{\phi}(\theta, x_0, \bar{p}) \right\}, \quad G(\theta + 2\pi) = G(\theta),$$

which nominally stabilizes System (3) (or System (35)) for $p = \bar{p}$ about the nominal periodic orbit $\mathcal{C}$. Substitute the control (36) in System (3) and linearize the closed-loop system about the true periodic solution $\tilde{\phi}$ to be well-suited for adaptation and in the following we suggest two strategies for adaptive control modified according to the latest estimate of the parameters. Event-Driven GSHF control turns out to be relatively insensitive to some online estimation process such as an Extended Kalman Filter and the controller is continuously updated.

2. We introduce adaptive control strategies where the uncertain parameters are estimated online by orbit by referencing the nominal periodic orbit. In this section, as a formal solution to Problem 2, we simulate the closed-loop system with the perturbed parameter $p$, which nominally stabilizes System (3) (or System (35)) for $p = \bar{p}$. Consequently, we anticipate that Event-Driven GSHF controllers are relatively insensitive to disturbance term in the linearized system (37). However, note that the true natural periodic solution $\tilde{\phi}(\theta, x_0^*(p), p)$ of System (35) has period $2\pi$ regardless of the value of $p$, moreover $\tilde{\phi}(\theta, x_0^*(p), p)$ and $\tilde{\phi}(\theta, x_0^*(\bar{p}), \bar{p})$ are not significantly different as long as $\mathcal{C}(p)$ is close to $\mathcal{C}$. Therefore, if $p$ is close to $\bar{p}$, then $A(\theta, p)$ and $B(\theta, p)$ are close to $A(\theta, \bar{p})$ and $B(\theta, \bar{p})$, respectively, and the disturbance term is small. Consequently, we anticipate that Event-Driven GSHF controllers are relatively insensitive to parameter perturbations compared to Time-Driven GSHF controllers.

**Example 2** (Reversed-Time Van der Pol System, Revisited). Consider again the Reversed-Time Van der Pol system from Example 1. This time, we shall nominally stabilize System (17) about the natural periodic orbit in Figure 2 using Event-Driven GSHF control. The map $\alpha$ is taken to be the identity, i.e., $[v_1, v_2]^T = \alpha([x, y]^T) = [x, y]^T$, and the control law is synthesized according to (27)-(34), where the weighting matrices are chosen as $Q = I$, $R' = I$, and $V' = 0$. The eigenvalues of the open-loop and closed-loop monodromy matrices are obtained as $\{51.35, 1.000\}$ and $\{0.997, 5.47 \times 10^{-6}\}$, respectively. A controlled trajectory of the nominal system with Event-Driven GSHF control was simulated and is shown in Figure 7. The initial state for the simulation is the same as in Example 1. It is seen that the controlled trajectory stays close to and converges to the natural periodic orbit and therefore nominal asymptotic stability about the orbit is achieved by Event-Driven GSHF control. (Compare Figure 7 with Figure 3.)

Next, in the same way as in Example 1 we simulated the closed-loop system with the perturbed parameter value, $p = 0.8$, using the same controller above. A simulated controlled trajectory with the perturbed parameter is shown in Figure 8. This time, in contrast to Example 1, the controlled trajectory remains close to the natural periodic orbit in spite of the parameter perturbation. (Compare Figure 8 with Figure 4.) The state, however, does not converge to the true periodic orbit shown in Figure 5 because of the disturbance term in (37).

**ADAPTIVE EVENT-DRIVEN GSHF CONTROL**

So far, we have seen that Event-Driven GSHF controllers generally have more robustness than Time-Driven GSHF controllers with respect to parameter perturbations. Under parameter perturbations, however, it is not possible for the system to converge to the true natural periodic orbit by referencing the nominal periodic orbit. In this section, as a formal solution to Problem 2, we introduce adaptive control strategies where the uncertain parameters are estimated online by some online estimation process such as an Extended Kalman Filter and the controller is continuously modified according to the latest estimate of the parameters. Event-Driven GSHF control turns out to be well-suited for adaptation and in the following we suggest two strategies for adaptive control using Event-Driven controllers.
Consider System (3) and the transformed System (35). We assume that the uncertain parameter $p$ is observable from the output of the system and let $p_{\text{est}}(t)$ be the estimate of $p$ at time $t$ by an online estimation process. Assume further that $p_{\text{est}}(t) \rightarrow p$ as $t \rightarrow \infty$.

The first strategy is to update the reference orbit only. Suppose we have designed a nominal stabilizing control as in (36). Using the estimate $p_{\text{est}}$ of $p$, we modify the control as follows:

$$u(\theta) = G(\theta) \left\{ x(\theta) - \hat{\phi}(\theta, x_0^*(p_{\text{est}}), p_{\text{est}}) \right\},$$

where $G(\theta)$ is the same as in (36). Substituting this control in System (35) and linearizing the closed-loop system about $\hat{\phi}(\theta, x_0^*(p), p)$, we obtain

$$\delta x' = \left[ A(\theta, p) + B(\theta, p)G(\theta) \right] \delta x + G(\theta) \left\{ \hat{\phi}(\theta, x_0^*(p), p) - \hat{\phi}(\theta, x_0^*(p_{\text{est}}), p_{\text{est}}) \right\}. \quad (40)$$

Since $p_{\text{est}}(t) \rightarrow p$ as $t \rightarrow \infty$, the disturbance term in System (40) vanishes as $t \rightarrow \infty$. Therefore, if System (40) is stable, then $x(\theta) \rightarrow \hat{\phi}(\theta, p)$ as $t \rightarrow \infty$, i.e., the controlled trajectory converges to the true periodic orbit asymptotically.

The second, more sophisticated strategy, is to update the feedback gain as well as the reference periodic orbit. The control therefore takes the form

$$u(\theta) = G(\theta, p_{\text{est}}) \left\{ x(\theta) - \hat{\phi}(\theta, x_0^*(p_{\text{est}}), p_{\text{est}}) \right\},$$

where $G(\theta, p_{\text{est}})$ is obtained such that the linear periodic system with period $2\pi$

$$\delta x' = \left[ A(\theta, p_{\text{est}}) + B(\theta, p_{\text{est}})G(\theta, p_{\text{est}}) \right] \delta x,$$

is stabilized using the technique in Appendix. With this control it is guaranteed that the system is stabilized about $\mathcal{C}(p)$ asymptotically.

In the adaptive control strategies above, we implicitly assumed that for each $p_{\text{est}}$ the new reference periodic orbit $\hat{\phi}(\theta, x_0^*(p_{\text{est}}), p_{\text{est}})$ is instantaneously available. But that may not always be a reasonable assumption since finding a periodic orbit usually involves many integrations of the nonlinear system. Therefore, we shall derive the first order approximation of $\hat{\phi}(\theta, x_0^*(p), p)$ for $p \neq \bar{p}$ about the nominal solution $\hat{\phi}(\theta, x_0^*(\bar{p}), \bar{p})$. For each $p$ we have, for a periodic solutions of System (35),

$$\hat{\phi}(2\pi, x_0^*(p), p) = x_0^*(p). \quad (43)$$
The first-order variation of (43) about $\phi(\theta, \bar{x}_0, \bar{p})$ is
\begin{equation}
\frac{\partial \tilde{\phi}}{\partial x_0} \delta x_0^* + \frac{\partial \tilde{\phi}}{\partial p} \delta p = \delta x_0^*,
\end{equation}
where $\delta x_0^* = x_0^*(p) - \bar{x}_0^*$, $\delta p = p - \bar{p}$, and the Jacobian matrices are evaluated at $(2\pi, \bar{x}_0^*, \bar{p})$. Solving (44) for $\delta x_0^*$, we obtain
\begin{equation}
\delta x_0^* = - \left( \frac{\partial \tilde{\phi}}{\partial x_0} - I \right)^{-1} \frac{\partial \tilde{\phi}}{\partial p} \delta p.
\end{equation}
Equation (45) gives the variation of the initial state $x_0^*(p)$ due to a variation of $p$. However, $\tilde{\phi}(\theta, \bar{x}_0^* + \delta x_0^*, \bar{p} + \delta p)$, where $\delta x_0^*$ is from (45) is not in general periodic because of the nonlinearity of System (35). Now, we expand $\tilde{\phi}(\theta, \bar{x}_0^* + \delta x_0^*, \bar{p} + \delta p)$ and neglect the higher order terms:
\begin{equation}
\tilde{\phi}(\theta, \bar{x}_0^* + \delta x_0^*, \bar{p} + \delta p) \approx \tilde{\phi}(\theta, \bar{x}_0^*, \bar{p}) + \Phi(\theta, 0) \delta x_0^* + S(\theta) \delta p,
\end{equation}
where
\begin{equation}
\Phi(\theta, 0) \triangleq \frac{\partial \tilde{\phi}}{\partial x_0}(\theta, \bar{x}_0^*, \bar{p}),
\end{equation}
\begin{equation}
S(\theta) \triangleq \frac{\partial \tilde{\phi}}{\partial p}(\theta, \bar{x}_0^*, \bar{p}).
\end{equation}
Substituting (45) in (46) and using (47)-(48), we obtain
\begin{equation}
\tilde{\phi}(\theta, \bar{x}_0^* + \delta x_0^*, \bar{p} + \delta p) \approx \tilde{\phi}(\theta, \bar{x}_0^*, \bar{p}) + E(\theta) \delta p,
\end{equation}
where
\begin{equation}
E(\theta) \triangleq S(\theta) - \Phi(\theta, 0) \left[ \Phi(2\pi, 0) - I \right]^{-1} S(2\pi).
\end{equation}
The right hand side of (49) is the first order approximation of $\tilde{\phi}(\theta, x_0^*(p), p)$ about the nominal periodic orbit. Observe that the first order approximation is also periodic with period $2\pi$ as desired. The state transition matrix $\Phi(\theta, 0)$ and the sensitivity matrix $S(\theta)$ about the nominal periodic orbit are obtained, respectively, by solving the following differential equations:
\begin{equation}
\Phi'(\theta, 0) = A(\theta, \bar{p}) \Phi(\theta, 0), \quad \Phi(0, 0) = I,
\end{equation}
\begin{equation}
S'(\theta) = A(\theta, \bar{p}) S(\theta) + \frac{\partial \tilde{f}}{\partial p}, \quad S(0) = 0.
\end{equation}
Thus, once $\Phi(\theta, 0)$ and $S(\theta)$ are computed and stored for over one period, the first order approximation of the natural periodic orbit is easily evaluated using the algebraic expressions (49)-(50) for any $p$ near $\bar{p}$. Note that the above simple result is only possible as a result of changing the independent variable of the system to $\theta$. This first-order approximation of the periodic orbits can be used together with the adaptive strategies (39) or (41). In so doing, we replace the updated reference periodic orbit by the first-order approximation:
\begin{equation}
\tilde{\phi}(\theta, x_0^*(p_{\text{est}}), p_{\text{est}}) \approx \tilde{\phi}(\theta, \bar{x}_0^*, \bar{p}) + E(\theta)(p_{\text{est}} - \bar{p}).
\end{equation}
It is also possible to obtain the second- or higher-order approximation of $\tilde{\phi}(\theta, x_0^*(p), p)$ in the same manner. These higher-order approximation may be used when the first-order approximation is not satisfactory.

**Example 3** (Reversed-Time Van der Pol System, Re-revisited). We consider the Reversed-Time Van der Pol system (17) one more time. We let $\bar{p} = 0.6$ be the nominal value of $p$ and $p = 0.8$ be the true value. We use the first adaptive control strategy (39) (updating only the reference orbit) together with Event-Driven GSHF control to robustly stabilize the system about the true natural
periodic orbit. The uncertain parameter $p$ is estimated online using an Extended Kalman Filter and the first-order approximation (53) is used for the updated reference periodic orbit. The design of Event-Driven GSHF controller is the same as in Example 2 and the eigenvalues of the closed-loop monodromy matrix are the same as well. A simulated controlled trajectory is shown in Figure 9. The time history of the estimate of $p$ is shown in Figure 10. The controlled trajectory stays close to and converges to a neighborhood of the true natural periodic orbit shown in Figure 5.

![Figure 9: Controlled trajectory of the perturbed Reversed-Time Van der Pol system by Adaptive Event-Driven GSHF control: $p = \bar{p} + 0.2 = 0.8$](image)

![Figure 10: Time history of the estimate of $p$ by Extended Kalman Filter for Adaptive Event-Driven GSHF control of the perturbed Reversed-Time Van der Pol system](image)

**ROBUST STABILIZATION OF A HALO ORBIT**

In this section, as an application of our method, we consider flight control of a spacecraft along an unstable Halo orbit in a general planet-satellite system where the satellite’s mass is uncertain. Uncertainties of planetary satellite masses are very common because there are many planetary satellites in the solar system whose mass estimates are only available with a few digits (See Ref. 21). The model we use here for the spacecraft’s motion is that of Hill’s three body problem which are very general (See Refs. 22–24). The basic Hill’s assumptions require a near circular orbit for the planetary satellite and a sufficiently small relative mass ratio of the satellite to the planet. These conditions are satisfied for almost all planetary satellites in the solar system, exceptions being Earth’s moon and Pluto’s moon, Charon, for which the mass ratios are not small.\(^{25}\)

The Hill’s equations of motion with added control inputs are given by

\[
\begin{align*}
\ddot{x} - 2\omega \dot{y} &= \frac{\partial V}{\partial x} + u_x, \\
\ddot{y} + 2\omega \dot{x} &= \frac{\partial V}{\partial y} + u_y, \\
\ddot{z} &= \frac{\partial V}{\partial z} + u_z,
\end{align*}
\]

where $V(x, y, z) = \mu \frac{1}{r} + \frac{1}{2} \omega^2 (3x^2 - z^2)$, $r = \sqrt{x^2 + y^2 + z^2}$,

\[
V(x, y, z) = \frac{\mu}{r} + \frac{1}{2} \omega^2 (3x^2 - z^2), \quad r = \sqrt{x^2 + y^2 + z^2},
\]

where $[x, y, z]^T$ describes the position of the spacecraft in a rotating Cartesian frame such that the origin coincides with the center of the satellite, the $x$-axis lies along the planet-satellite line pointing outward, the $y$-axis lies along the motion of the satellite about the planet, and the $z$-axis completes the right-handed triad, $\omega$ is the satellite’s rotation rate about the planet assuming circular motion, and $\mu$ is the satellite’s mass parameter. The value of the mass parameter $\mu$ is assumed to be uncertain with the nominal value $\bar{\mu}$. In order to avoid using specific values for $\omega$ and $\bar{\mu}$, it is possible
to remove those parameters from the equations by scaling time by $1/\omega$ and length by $(\bar{\mu}/\omega^2)^{1/3}$. The resulting dimensionless equations are

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} + u_x, \quad (58)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} + u_y, \quad (59)$$

$$\ddot{z} = \frac{\partial U}{\partial z} + u_z, \quad (60)$$

$$U(x, y, z) = \frac{\sigma}{r} + \frac{1}{2}(3x^2 - z^2), \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (61)$$

where $\sigma \triangleq \mu/\bar{\mu}$ is the ratio of the true value to the nominal value of the mass parameter. We shall use these dimensionless equations and assume that $\sigma$ is an uncertain parameter with the nominal value 1.

We let $\dot{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T$ be the state vector and $u = [u_x, u_y, u_z]^T$ be the control vector. The Hill system (58)-(60) can be written in standard first-order form as

$$\dot{x} = f(x, u, \sigma) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ U_x + 2\dot{y} + u_x \\ U_y - 2\dot{x} + u_y \\ U_z + u_z \end{bmatrix}, \quad (62)$$

where $U_x = \partial U/\partial x$, $U_y = \partial U/\partial y$ and $U_z = \partial U/\partial z$. The nominal system $\dot{x} = f(x, u, 1)$ has an unstable natural periodic orbit, called a Halo orbit, near the $L_2$ libration point shown in Figure 11. This Halo orbit is obtained with the initial condition below.

$$x(0) = 0.7719096203522987 \quad \dot{x}(0) = -3.506665977700358 \times 10^{-14}$$

$$y(0) = 0.0000000000000000 \quad \dot{y}(0) = -0.6528955641722595$$

$$z(0) = 0.135324498885092 \quad \dot{z}(0) = -1.213517505267137 \times 10^{-13} \quad (63)$$

Note that specification of all the digits in the above data is required because of the severe instability. The period of the nominal periodic orbit is $T \approx 3.0781$. The linearized system about the nominal periodic solution is obtained as

$$\dot{\delta x} = A\delta x + Bu, \quad (64)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & U_{xz} & 0 & 2 & 0 \\ U_{yx} & U_{yy} & U_{yz} & -2 & 0 & 0 \\ U_{zz} & U_{zy} & U_{zz} & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (65)$$

where $U_{xx} = \partial^2 U/\partial x^2$, $U_{xy} = \partial^2 U/\partial x \partial y$, etc, are evaluated along the nominal periodic solution. The characteristic multipliers of the periodic orbit, i.e., the eigenvalues of the open-loop monodromy matrix $\Phi(T, 0)$ computed by integrating

$$\Phi(t, 0) = A\Phi(t, 0), \quad \Phi(0, 0) = I, \quad (66)$$

over one period, are

$$1563, 1/1563, 0.980 \pm 0.197i, 1.000, 1.000,$$
and the first eigenvalue indicates that the Halo orbit is in fact a very unstable periodic orbit. It is easily checked that \( \sigma \) is observable through the state, implying that navigation can be used to estimate the value of \( \sigma \).

First, we shall stabilize the nominal system \( \dot{x} = f(x, u, 1) \) about the nominal periodic orbit by Time-Driven GSHF control. We choose the weighting matrices \( Q' = I \), \( R' = 10^{-2} \cdot I \), \( V' = 0 \) and synthesize the controller as in (9)-(16). The eigenvalues of the resulting closed-loop monodromy matrix are

\[
0.0333, \quad 0.00409 \pm 0.108i, \quad 1.48 \times 10^{-10}, \quad 7.26 \times 10^{-4}, \quad 0.00540,
\]

indicating Lyapunov asymptotic stability. Figure 12 shows a simulated controlled trajectory of the nominal system with this Time-Driven GSHF control. The initial state for the controlled trajectory is set at the point where \( x(0) \) is shifted by 0.0463 from the initial state for the nominal periodic trajectory in (63). (The following series of simulations are obtained with the same initial state for comparison.) We also simulated the closed-loop system with the perturbed parameter value \( \sigma = 1.05 \), which corresponds to a +5% perturbation of the mass parameter \( \mu \) from the nominal value \( \bar{\mu} \), using the same Time-Driven GSHF controller. Figure 13 shows a simulated controlled trajectory in this case. It is seen that the controller fails to keep the spacecraft in the vicinity of the periodic orbit in this case as a result of the parameter perturbation.

Next, we apply Adaptive Event-Driven GSHF control to the spacecraft flight. We take the

Figure 12: Controlled trajectory of the nominal Hill system by Time-Driven GSHF control
map $\alpha$ to be an $xy$-projection such that

$$[v_1, v_2]^T = \alpha([x, y, z, \dot{x}, \dot{y}, \dot{z}]^T) = [y, x - c]^T,$$

where $c = 0.65$. By changing the independent variable from time to $\theta$, (62) is transformed to

$$\frac{dx}{d\theta} = f(x, u, \sigma) = \frac{(x - c)^2 + y^2}{xy - \dot{y}(x - c)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ U_x + 2\dot{y} + u_x \\ U_y - 2\dot{x} + u_y \\ U_z + u_z \end{bmatrix},$$

Then, we choose the weighting matrices $Q' = I$, $R' = 10^{-2} \cdot I$, $V' = 0$ and synthesize an Event-Driven GSHF controller according to (27)-(34). We apply the first adaptive strategy (39) where the controller updates only the reference periodic orbit and uses the first order approximation (53) for the updated periodic orbits. In the following simulations we assume immediate estimation of the parameter $\sigma$, i.e. $\sigma_{est}(0) = 1$ and $\sigma_{est}(t) \equiv \sigma$ for $t > 0$. This assumption is reasonable because the time constant of an estimator would typically be many orders of magnitude shorter than the period of the Halo orbit. The eigenvalues of the closed-loop monodromy matrix $\Psi(2\pi)$ in (30) are

$$0.997, 0.0284, 0.0242, 0.0115, 2.25 \times 10^{-10}, 0.00177$$

indicating orbital asymptotic stability. Figure 14 shows a simulated controlled trajectory of the nominal Hill system by (Adaptive) Event-Driven GSHF control. Figure 15, 16, 17, 18 show simulated controlled trajectories by Adaptive Event-Driven GSHF control with perturbed parameter values $\sigma = 1.05, 0.95, 1.10, 0.90$, respectively. The trajectories were simulated for $5T$. In these simulations the controlled trajectory is seen to robustly stay close to and converge to the neighborhood of its natural periodic orbit. Figure 19 shows the first-order approximations of perturbed periodic orbits computed by (49) for $\sigma = 1 \pm 0.05, 1 \pm 0.10$.

**CONCLUSIONS**

We considered the problem of robust stabilization of nonlinear systems containing uncertain parameters about unstable periodic orbits and presented extensions of Generalized Sampled-data Hold Function (GSHF) control to address this problem. Time-Driven GSHF control can locally stabilize a nominal system about a nominal natural periodic orbit, but can have poor robustness
Figure 14: Controlled trajectory of *nominal* Hill system by (adaptive) Event-driven GSHF control: $\sigma = 1$

Figure 15: Controlled trajectory of *perturbed* Hill system by adaptive Event-driven GSHF control: $\sigma = 1.05$

Figure 16: Controlled trajectory of *perturbed* Hill system by adaptive Event-Driven GSHF control: $\sigma = 0.95$
Figure 17: Controlled trajectory of perturbed Hill system by adaptive Event-Driven GSHF control: $\sigma = 1.10$

Figure 18: Controlled trajectory of perturbed Hill system by adaptive Event-Driven GSHF control: $\sigma = 0.90$

Figure 19: First-order approximation of perturbed periodic orbits for $\sigma = 0.90, 0.95, 1.00, 1.05, 1.10$
with respect to parametric uncertainties in the system. This is due to changes in location and period of the natural periodic orbit by parameter changes. Period changes may result in large errors between the nominal periodic solution and the actual periodic solution even if parameter errors are relatively small. First, uncertainty in orbit’s period is addressed by making GSHF control “event-driven” rather than “time-driven”, which is simply done by changing the independent variable from time to a new angular variable. Event-Driven GSHF control is proven to have more robustness with respect to parameter perturbations than Time-Driven GSHF control. Second, uncertainty in orbit location is addressed by making Event-Driven GSHF control adaptive so that the reference periodic orbit is continuously updated. This adaptive control strategy further improves robustness and performance of the control law. The presented method was applied to flight control of a spacecraft about an unstable Halo orbit and showed good robustness in stabilization with respect to mass parameter uncertainty.

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REFERENCES


APPENDIX: GSHF CONTROL OF LINEAR PERIODIC SYSTEMS

In this appendix we review how Generalized Sampled-Data Hold Function control can be used to stabilize linear periodic systems. The results in this appendix are mostly given in Ref. 1. In addition to the basic review, we introduce a specific design of the hold function.

GSHF Basics

Consider the continuous-time, linear, periodic, state-space system

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad (69) \]

\[ A(t + T) = A(t), \quad B(t + T) = B(t), \quad \text{for all } t \geq 0, \quad (70) \]

where \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in \mathbb{R}^m \) is the control vector. The matrices \( A \) and \( B \) are periodic in \( t \) with period \( T \). Suppose that we want to find a state feedback control law \( u(t) = k(t, x(t)) \) that stabilizes System (69).

*Generalized Sampled-Data Hold Function (GSHF) control* has the form

\[ u(t) = H(t)x(kT), \quad t \in [kT, (k+1)T), \quad k = 0, 1, 2, \ldots, \quad (71) \]
\[ H(t + T) = H(t). \]  

(72)

Let \( \Phi(t, t_0) \) be the (open-loop) state transition matrix of System (69), which is obtained by solving

\[
\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I.
\]

(73)

The periodicity of \( A(t) \) and (73) imply

\[ \Phi(t + T, t_0 + T) = \Phi(t, t_0), \]

(74)

for all \( t, t_0 \). The Variation of Constants formula applied to System (69) gives

\[
x((k+1)T) = \Phi((k+1)T, kT)x(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)u(\tau)d\tau,
\]

(75)

and

\[
x(kT + \sigma) = \Phi(kT + \sigma, kT)x(kT) + \int_{kT}^{kT+\sigma} \Phi(kT + \sigma, \tau)B(\tau)u(\tau)d\tau, \quad \sigma \in [0, T),
\]

(76)

for \( k = 0, 1, 2, \ldots \). Substituting (71) in (75) and (76), and using (70) and (74), we obtain

\[
x((k+1)T) = \Psi(T)x(kT),
\]

(77)

and

\[
x(kT + \sigma) = \Psi(\sigma)x(kT), \quad \sigma \in [0, T),
\]

(78)

for \( k = 0, 1, 2, \ldots \), where

\[
\Psi(t) \triangleq \Phi(t, 0) + \int_0^t \Phi(t, \tau)B(\tau)H(\tau)d\tau, \quad t \in [0, T].
\]

(79)

Equation (77) is a discrete-time linear time-invariant system that governs transition of the state of the closed-loop system at discrete time \( t = kT, \ k = 0, 1, 2, \ldots \). The stability of the closed-loop system is determined by that of the discrete-time system (77). Thus, the closed-loop system is asymptotically stable if and only if all eigenvalues of \( \Psi(T) \) have moduli strictly less than one.

If in (71) we let \( H(t) \) have the form

\[
H(t) = F(t)K,
\]

(80)

where \( F(t) \) is periodic with period \( T \) and \( K \) is constant, then \( \Psi(T) \) can be written as

\[
\Psi(T) = \Phi(T, 0) + W(T)K,
\]

(81)

where

\[
W(t) \triangleq \int_0^t \Phi(t, \tau)B(\tau)F(\tau)d\tau, \quad t \in [0, T].
\]

(82)

Therefore, in this case, the eigenvalues of \( \Psi(T) \) are arbitrarily assignable by choice of the constant matrix \( K \) if, and only if, the pair \( (\Phi(T, 0), W(T)) \) is controllable. Note that the sizes of matrices of \( F(t) \) and \( K \) are yet to be specified as long as \( F(t)K \) is \( p \times n \).

In practice, GSHF control in the original form (71) is not advisable because, since the system evolves open-loop between sampling moments, it yields poor disturbance rejection and robustness with respect to model errors. But, using (78), the GSHF control (71) can be implemented as a continuous feedback control

\[
u(kT + \sigma) = H(\sigma)\Psi(\sigma)^{-1}x(kT + \sigma), \quad \sigma \in [0, T), \ k = 0, 1, 2, \ldots,
\]

(83)
whenever $\Psi(\sigma)$ is nonsingular. Note that the matrix $\Psi(\sigma)$ can be obtained by solving

$$\dot{\Psi}(\sigma) = A(\sigma)\Psi(\sigma) + B(\sigma)H(\sigma), \quad \Psi(0) = I.$$  \hspace{1cm} (84)

It is preferable in practice that GSHF control be implemented as a continuous feedback control in the form of (83). It is interesting to notice that the control in (83) has the form of continuous periodic-gain state feedback:

$$u(t) = G(t)x(t),$$  \hspace{1cm} (85)

$$G(t + T) = G(t), \text{ for all } t \geq 0.$$  \hspace{1cm} (86)

## Hold Function Design

The problem of stabilizing the periodic system (69)-(70) has been reduced to that of designing appropriate matrices $F(t)$ and $K$. Although there can be many choices of these matrices that give stability and acceptable performance, we next present the specific design for these matrices that we have used in this paper.

When $F(t)$ is specified, a stabilizing constant matrix $K$ can be obtained by Linear Quadratic Regulator design as follows. Consider the quadratic performance index

$$J = \int_0^\infty \{x(t)^TQx(t) + u(t)^TRu(t)\} dt,$$  \hspace{1cm} (87)

where $Q$ and $R$ are the symmetric positive definite weighting matrices. When GSHF control is applied, this continuous-time performance index can be transformed into the discrete-time quadratic performance index

$$J = \sum_{k=0}^{\infty} x(kT)^T(Q' + K^TR'K + 2V'K)x(kT),$$  \hspace{1cm} (88)

with

$$Q' = \int_0^{T} \Phi(\sigma,0)^TQ\Phi(\sigma,0)d\sigma,$$  \hspace{1cm} (89)

$$R' = \int_0^{T} W(\sigma)^TQW(\sigma) + F(\sigma)^TRF(\sigma)d\sigma,$$  \hspace{1cm} (90)

$$V' = \int_0^{T} \Phi(\sigma,0)^TQW(\sigma)d\sigma.$$  \hspace{1cm} (91)

The matrix $K$ that minimizes $J$ in (88) is given by

$$K = -\left(W(T)^TSW(T) + R'\right)^{-1}\left(W(T)^T S\Phi(T,0) + V'^T\right),$$  \hspace{1cm} (92)

where $S$ is the positive semi-definite solution of the Algebraic Riccati Equation

$$\Phi(T,0)^T S \Phi(T,0) - S + Q'$$

$$- \left(\Phi(T,0)^T SW(T) + V'\right) \left(W(T)^T SW(T) + R'\right)^{-1}\left(W(T)^T S\Phi(T,0) + V'^T\right) = 0.$$  \hspace{1cm} (93)

If the pair $(\Phi(T,0),W(T))$ is stabilizable, $R' > 0$, $Q' - V'R'^{-1}V'^T \geq 0$, and

$$(Q' - V'R'^{-1}V'^T, \Phi(T,0) - W(T)R'^{-1}V'^T)$$

has no unobservable mode on the unit circle, then it is guaranteed that this $K$ makes $\Psi(T)$ asymptotically stable.
While the constant matrix $K$ determines the behavior of the closed-loop system at sampling moments, the periodic matrix $F(t)$ influence the inter-sampling behavior of the closed-loop system governed by (78). Preferably, $F(t)$ should be chosen such that an induced norm $\|\Psi(t)\|_p$ is as small as possible for $t \in [0, T]$ for good performance. However, designing $F(t)$ as such is not a trivial problem because $\Psi(t)$ depends on both $F(t)$ and $K$, while $K$ is chosen after $F(t)$ is specified. In our experience, however, the choice

$$F(t) = B(t)^T \Phi(T, t)^T, \ t \in [0, T),$$

(94)

seems to work very well in many cases. Note that with this $F(t)$,

$$W(T) = \int_0^T \Phi(T, \sigma)B(\sigma)B(\sigma)^T\Phi(T, \sigma)^T d\sigma$$

(95)

coincides with the reachability grammian from $t = 0$ to $t = T$. 

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